Lecture 20. Methods for solving the radiative transfer equation. Part 3. Discrete-ordinate method.

Objectives:

1. Discrete-ordinate method for the case of isotropic scattering.

2.Generalization of the discrete-ordinate method for inhomogeneous atmosphere.

3. Numerical implementation of the discrete-ordinate method: DISORT

Required reading:

L02: 6.2

Recommended reading

Thomas G.E. and K. Stamnes, Radiative transfer in the atmosphere and ocean, 2000,

Chapter 8.1-8.10

1. Discrete-ordinate method for the case of isotropic scattering.

- A discrete-ordinate method has been developed by Chandrasekhar in about 1950 (see Chandrasekhar S., Radiative transfer, 1960, Dover Publications).

Recall the radiative transfer equation for azimuthally independent diffuse intensity:

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')d\mu' - \frac{\omega_0}{4\pi}F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)$$

For isotropic scattering, the scattering phase function is 1. Hence we have

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')d\mu' - \frac{\omega_0 F_0}{4\pi}\exp(-\tau/\mu_0) \qquad [20.1]$$

Let's apply the Gauss formula to replace the integral in Eq.[20.1]

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{j=-n}^{n} a_j I(\tau,\mu_j) - \boxed{\frac{\omega_0 F_0}{4\pi}\exp(-\tau/\mu_0)} \qquad [20.2]$$

Inhomogeneous part

where $i=-n, \ldots, n$ (2n terms) and $\boldsymbol{a_j}$ are the Gaussian weights (constants) and $\boldsymbol{\mu_j}$ are quadrature angles (or points).

Eq.[20.2] is a system of 2n inhomogeneous differential equations:

Solution of Eq.[20.2] = general solution + particular solution

where the general solution is a solution of the homogeneous part of the Eq.[20.2]

Denoting $\boldsymbol{I_i} = \boldsymbol{I_i}\ (\boldsymbol{\tau}, \boldsymbol{\mu_i})$, the general solution of Eq.[20.2] can be found as

$$I_i = g_i \exp(-k\tau) \tag{20.3}$$

Inserting Eq.[20.3] into Eq.[20.2], we obtain

$$g_i(1 + \mu_i k) = \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j g_j \tag{20.4}$$

We can find $\boldsymbol{g_i}$ in the form

$$g_i = L/(1 + \mu_i k)$$

where L is a constant to be determined. Substituting this expression for $\boldsymbol{g_i}$ in Eq.[20.4], we have

$$1 = \frac{\omega_0}{2} \sum_{j=-n}^{n} \frac{a_j}{1 + \mu_i k} = \omega_0 \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2} \tag{20.5}$$

Eq.[20.5] gives 2n solutions for +/-k_j (j=1,…,n).

Thus general solution is

$$I_i = \sum_j \frac{L_j}{1+\mu_i k_j} \exp(-k_j \tau) \qquad [20.6]$$

where $\boldsymbol{L_j}$ are constants.

The particular solution can be found as

$$I_i = \frac{\omega_0 F_0}{4\pi} h_i \exp(-\tau / \mu_0) \qquad [20.7]$$

where $\boldsymbol{h_i}$ are constants.

Inserting Eq.[20.7] into Eq.[20.2], we have

$$h_i (1 + \mu_i / \mu_0) = \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j h_j + 1 \qquad [20.8]$$

From Eq.[20.8], $\boldsymbol{h_i}$ is found as

$$h_i = \gamma / (1 + \mu_i / \mu_0)$$

where γ is determined from

$$\gamma = 1 / \{1 - \frac{\omega_0}{2} \sum_{j=1}^{n} a_j / (1 - \mu_j^2 / \mu_0^2)\} \qquad [20.9]$$

Adding the general solution Eq.[20.6] and the particular solution Eq.[20.7], we have the solution

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 \gamma}{4\pi (1 + \mu_i / \mu_0)} \exp(-\tau / \mu_0) \qquad [20.10]$$

where L_j are constants to be determined from the boundary conditions.

H-function has been introduced by Chandrasekhar as

$$H(\mu) = \frac{1}{\mu_1 \ldots \mu_n} \frac{\prod_{j=1}^{n} (\mu + \mu_j)}{\prod_{j=1}^{1} (1 + k_j \mu)} \qquad [20.11]$$

One can express $\boldsymbol{\gamma}$ in the H-function that Eq.[20.10] becomes

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 H(\mu_0) H(-\mu_0)}{4\pi (1 + \mu_i / \mu_0)} \exp(-\tau / \mu_0) \qquad [20.12]$$

Eq.[20.12] gives a simple solution for the semi-infinite isotropic atmosphere (see L02:6.2.2)

$$I^{\uparrow}(0, \mu) = \frac{1}{4\pi} \omega_0 F_0 \frac{\mu_0}{\mu + \mu_0} H(\mu_0) H(\mu) \qquad [20.13]$$

2. Generalization of the discrete-ordinate method for inhomogeneous atmosphere.

Let's consider the atmosphere with non-isotropic scattering.

We can expand the diffuse intensity in the cosine series

$$I(\tau,\mu,\varphi) = \sum_{m=0}^{N} I^{m}(\tau,\mu)\cos(m(\varphi_0-\varphi))$$

So we need to solve

$$\mu \frac{dI^{m}(\tau,\mu)}{d\tau} = I^{m}(\tau,\mu) - (1+\delta_{0,m})\frac{\omega_0}{4}\sum_{l=m}^{N} \varpi_l^m P_l^m(\mu)\int_{-1}^{1} P_l^m(\mu')I^m(\tau,\mu')d\mu' -$$

$$-\frac{\omega_0}{4\pi}\sum_{l=m}^{N} \varpi_l^m P_l^m(\mu)P_l^m(-\mu_o)F_0 \exp(-\tau/\mu_0)$$

The **general solution** may be written

$$I^m(\tau,\mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_j)\exp(-k_j^m \tau)$$

ϕ_j^m, k_j^m, L_j^m are coefficients to be determined.

The **particular solution** may be written

$$I_p^m(\tau,\mu_i) = Z^m(\mu_i)\exp(-\tau/\mu_0)$$

$Z^m(\mu_i)$ is a function

$$Z^m(\mu_i) = \frac{1}{4\pi}\omega_0 F_0 P_m^m(-\mu_0)\frac{H^m(\mu_0)H^m(-\mu_0)}{1+\mu_i/\mu_0}\sum_{l=0}^{N}\varpi_l^m \zeta_l^m \frac{1}{\mu_0}P_l^m(\mu_i)$$

The **complete solution** of the radiative transfer is

$$I^m(\tau,\mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_j)\exp(-k_j^m \tau) + Z^m(\mu_i)\exp(-\tau/\mu_0) \qquad [20.14]$$

i=-n,…,n

NOTE: If a layer has gases, aerosols and/or clouds, one needs to calculate the effective optical properties of this layer.

Let's generalize the **complete solution** Eq.[20.14] of the radiative transfer for the inhomogeneous atmosphere. The atmosphere can be divided into the N homogeneous layers, each is characterized by a single scattering albedo, phase function, and optical depth.

For ***l***-th layer, we can write the solution using Eq.[20.14]. To simplify notations, let's consider the azimuthal independent case (i.e., ***m=0***), so we have

$$I^{l}(\tau,\mu_i) = \sum_{j=-n}^{n} L_j^l \phi_j^l(\mu_j)\exp(-k_j^l\tau) + Z^l(\mu_i)\exp(-\tau/\mu_0) \qquad [20.15]$$

Now, we need to match the boundary and continuity conditions between layers.

At the top of the atmosphere (TOA): no downward diffuse intensity

$$I^{l=1}(0,-\mu_i) = 0 \qquad [20.16]$$

At the layer's boundary: upward and downward intensities must be continuous

$$I^{l}(\tau_l,\mu_i) = I^{l+1}(\tau_l,\mu_i) \qquad [20.17]$$

At the bottom of the atmosphere (assuming the Lamdertian surface):

$$I^{l=N}(\tau_N,\mu_i) = \frac{r_{sur}}{\pi}[F^{\downarrow}(\tau_N) + \mu_0 F_0\exp(-\tau_N/\mu_0)] \qquad [20.18]$$

Eqs.[20.16]-[20.18] provide necessary equations to find the unknown coefficients.

3. Numerical implementation of discrete-ordinate methods: DISORT.

DISORT is a FORTRAN numerical code based on the discrete-ordinate method developed by Stamnes, Wiscombe et al. DISORT is openly available and has a good user-guide.

1) DISORT applies to the inhomogeneous nonithothermal plane-parallel atmosphere.

2) A user may set-up any numbers of the plane-parallel layers.

3) Each layer must be characterized by the effective optical depth, single scattering albedo and asymmetry parameter if the Henyey-Greenstein phase function is used.

4) A user may use any phase function by providing the Legendre polynomial expansion coefficients.

5) A user selects a number of streams (keeping in mind that the computation time varies as n^3).

6) A key problem is to obtain a solution for fluxes for strongly forward-peaked scattering.

7) DISORT allows predicting the intensity as a function of the direction and position at any point in the atmosphere (i.e., not only at the boundaries of the layers).

Eddington Second Approximation for Radiances

The Eddington approximation can give accurate radiances only through a two-step process:

1) The Eddington solution gives the crude radiance field,
2) The source function, with scattering, is found from the Eddington I_0,I_1, and then integrated for the radiance.

Used for thermal emission with scattering, where source function is

$$J(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} P(\mu, \mu') I(\tau, \mu') d\mu' + (1 - \omega) B(T)$$

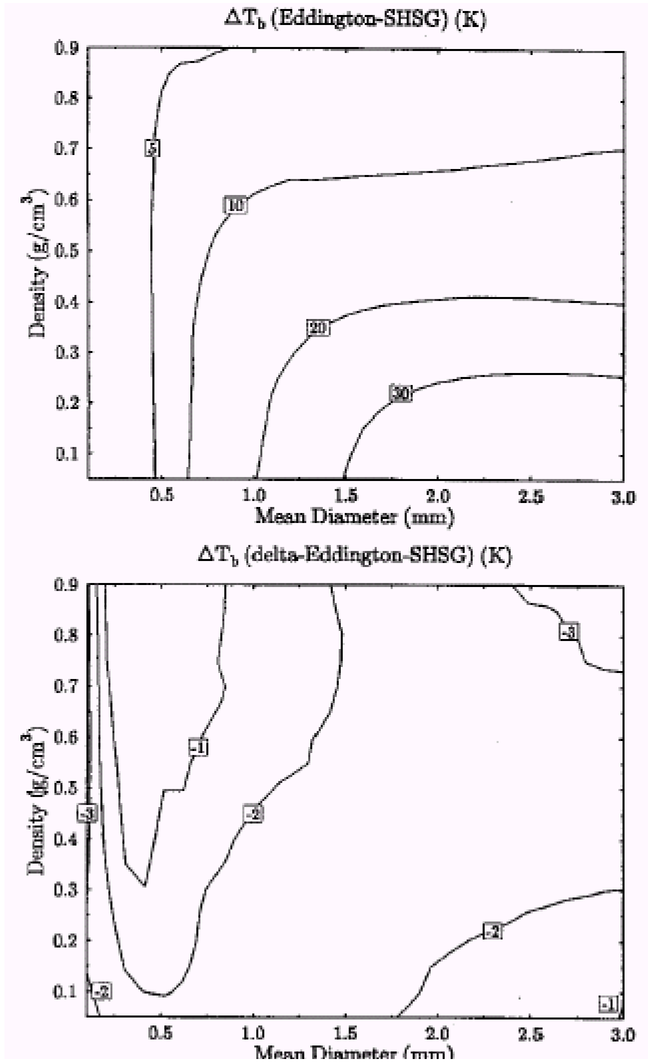
Putting in the Eddington approximation:

$$J_{edd}(\mu) = \omega(I_0 + I_1 g\mu) + (1 - \omega) B(T)$$

Upwelling radiance at top is then

$$I^{\uparrow}(\tau = 0, \mu) = \int_0^{\tau_*} J_{edd}(\tau, \mu)\, e^{-\tau/\mu}\, d\tau/\mu$$

Eddington's second approximation works well when scattering integral is like a low order moment.



Difference in upwelling zenith brightness temperature between the Eddington approximation and spherical harmonic ($L = 11$) radiative transfer methods at 85.5 GHz for a modeled ice particle layer. There is a single uniform ice sphere layer of optical depth 2 at 85.5 GHz with temperature from 270 to 245 K above a blackbody surface at 270 K. The top panel is for unscaled Eddington, and the bottom is for delta-scaled Eddington. [Evans, 1993, PhD thesis]

Multiple Scattering Flux Reflection Results

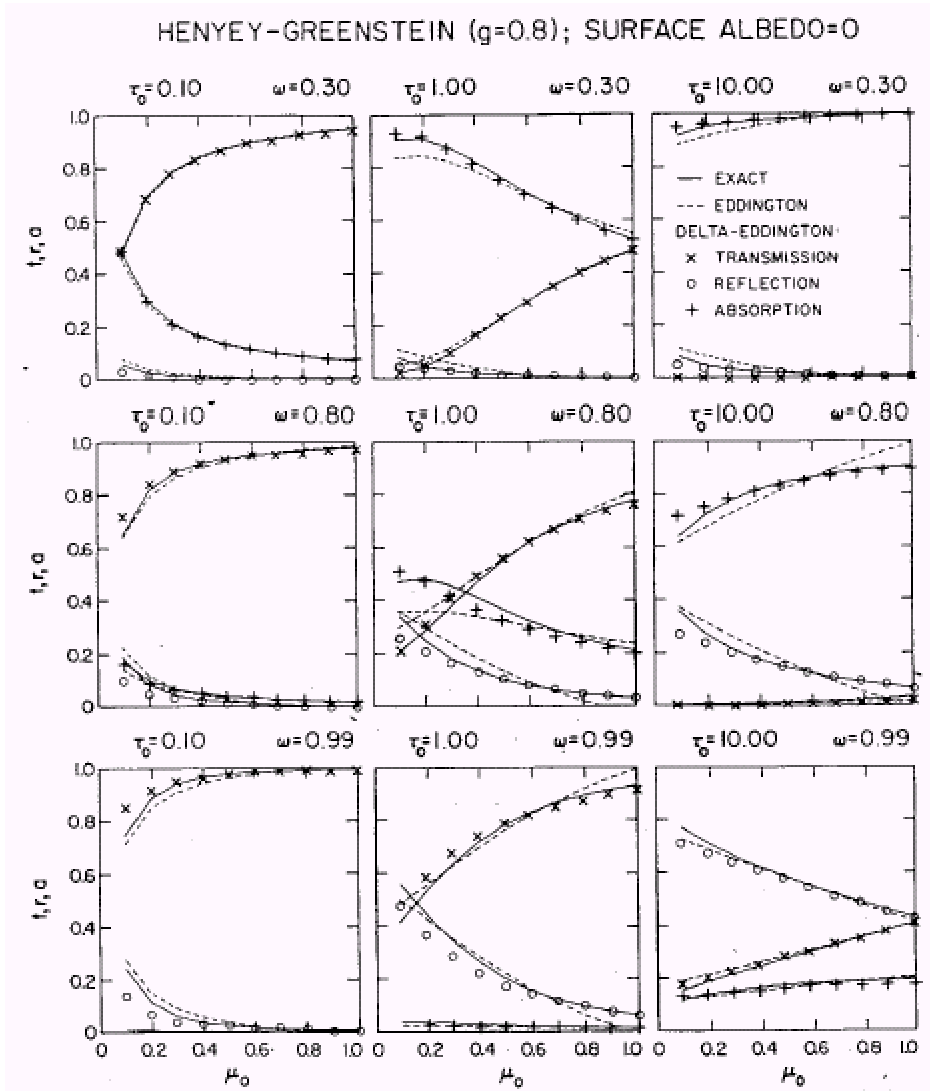
Fundamental property of reflectivity from radiative transfer:
- Linear for $\tau \ll 1$ (first order solution)
- Saturation for $\tau \gg 1$

More forward scattering means less reflection ($g \uparrow \ \Rightarrow \ R \downarrow$)
Equivalent isotropic scattering optical depth: $\tau' = (1 - \omega g)\tau$

Higher solar zenith angle means more reflection unless optically thin: ($\mu_0 \downarrow \ \Rightarrow \ R \uparrow$)

Multiple scattering amplifies absorption ($\mu_0 = 2/3 \;\; g = 0.85$):

$\tau = 1 \quad \omega = 0.99 \qquad R = 0.096 \; A = 0.018$
$\tau = 10 \quad \omega = 0.99 \qquad R = 0.45 \;\; A = 0.18$
$\tau = 100 \;\; \omega = 0.99 \qquad R = 0.55 \;\; A = 0.45$
$\tau = 100 \;\; \omega = 1.00 \qquad R = 0.92 \;\; A = 0.00$



Reflectivity (r), transmissivity (t), and absorptivity (a) as a function of cosine of solar zenith angle (μ_0) for various single-scattering albedoes (ω) and later optical depths (τ_0), comparing exact, Eddington and delta-Eddington methods for asymmetry factor $g = 0.8$ and surface albedo ($A = 0$). [Joseph and Wiscombe, 1976: The Delta-Eddington Approximation for Radiative Flux Transfer, J. Atmos. Sci., 33, 2452.]